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A PROOF OF THE THEOREM OF AMIR AND LINDENSTRAUSS

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ABSTRACT

We give a very short proof of the theorem concerning long strings of projections mentioned in the title. The nature of the proof is such that, for example, a result of Gul'ko follows easily.

The following result is due to Amir and Lindenstrauss [AL] and in this form to Vasak [V]; see also [Gu], [M], [NW], [T] and [P]. The proof is our own, very easy and self contained except for a few very easy results in topology (see [B] and [En]) and classical facts about commutative Banach algebras ([N]). Other than the Stone–Weierstraß theorem, which we often make use of without explicitly stating, we require the following facts which are very special cases of variants of theorems of Banach, Stone, Naimark and Gelfand: a uniformly closed subalgebra of $C(K)$ is closed in the simple topology and if P is a contractive projection on $C(K)$ ($P^2 = P$ and $\| P \| = 1$) so that $P(1_K) = 1_K$ and P is also an algebraic homomorphism then P is defined by a retraction on K, which means that there exists a continuous function $r : K \rightarrow K$ so that $r^2 = r$ and $P(f) = f \circ r$ for all f in $C(K)$. We make a quite elementary observation at this point. Suppose that X is a Banach space and $H \subseteq G \subseteq X^*$. Then H is weak* dense in G if and only if $H \mid F$ is dense in $G \mid F$ for each finite dimensional subspace F of X. We denote by den T the topological density of T, that is the minimal cardinality of a dense subset of T. In what follows K will denote a compact Hausdorff space and $C_s(K)$ the algebra of continuous functions on K in the topology s of pointwise convergence on K; s is also called

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the simple topology. We denote by nden T the minimum cardinality of a norm dense subset of T when T is a subset of a Banach space. We consider only Hausdorff spaces. Suppose that $\Phi : T \to \mathfrak{p}(S)$ is an upper semicontinuous compact valued map. The graph

$$G = \{(t, s) : t \in T \text{ and } s \in \Phi(t)\}$$

of Φ is a closed subset of $T \times S$. Let E be a closed subspace of G. Then

$$T_1 = \{t \in T\text{: there exists } s \in S \text{ so that } (t, s) \in E\}$$

is a closed subset of T and $\Phi_1 : T_1 \to \varphi(S)$ defined by $\Phi_1(t) = \{s \in S : (t, s) \in E\}$ is an upper semicontinuous compact valued map whose graph is E. Suppose that $f : S \to R$ is a continuous (single valued) function. Since $f \circ \Phi_1$ is also an upper semicontinuous compact valued map its graph is a closed subspace of $T_1 \times R$, which is a closed subspace of $T \times R$. This proves that the function from the graph G of Φ to the graph of $f \circ \Phi$ defined by mapping (t, s) to $(t, f(s))$ is a closed mapping.

In the result below, if we replace a separable M by an arbitrary metric space we obtain an analogous result except that the best that can be said is that

$$\text{nden}\, P(C(K)) = \max\{\text{nden}\, Y, \text{den}\, M\}.$$

Main Theorem. *Suppose that K is compact, M is a separable metric space, Φ an s upper semicontinuous compact valued map from M into $C(K)$ whose image $T = \bigcup_{m \in M} \Phi(m)$ separates the points of K and S is an arbitrary subset of $C(K)$. Then there exists a projection P defined on $C(K)$ such that P is defined by a retraction on K, $S \subseteq P(C(K))$,* nden $P(C(K)) = \max\{\omega, \text{nden}\, S\}$ *and $P(T) \subseteq T$.*

Proof. Let $G \subseteq C(K) \times M$ be the graph of Φ. For each finite $E \subseteq K$ choose a countable subset $G_E \subseteq G$ such that G_E is dense in G when both are restricted to $E \times M$. By this, we mean that given the canonical operator $U_E : C(K) \to l_\infty(E)$, then $(U_E \times I_M)(G_E)$ is dense in $(U_E \times I_M)(G)$. Let $\beta = \max\{\omega, \text{nden}\, S\} \leqq \text{den}\, C(K)$. Let A_1 be the smallest (norm closed and containing the constants) algebra that contains S. It follows that nden $A_1 \leqq \beta$. Choose any $\Gamma_1 \subseteq K$ of cardinality β that norms A_1. Let A_2 be the smallest algebra that contains

$$A_1 \cup \left(\bigcup_{E \subseteq \Gamma_1} \text{proj}\, G_E \right).$$

Then nden $A_2 \leqq \beta$. Choose $\Gamma_1 \subseteq \Gamma_2$ that norms A_2 and card $\Gamma_2 \leqq \beta$. Continue this for each positive integer. Let

$$\Gamma_S = \bigcup_n \Gamma_n \quad \text{and} \quad A_S = \overline{\bigcup_n A_n}^{\text{norm}} .$$

Define

$$G_1 = \overline{\bigcup \{G_E : E \subseteq \Gamma_S\}}$$

where the closure is taken in $C_s(K) \times M$. Denote by U the canonical operator from $C(K)$ to $l_\infty(\Gamma_S)$ and we assume that $l_\infty(\Gamma_S)$ has the simple topology. Note that $U \times I_M$ restricted to G is a closed mapping from G onto a closed subspace of $l_\infty(\Gamma_S) \times M$, G_1 is a closed subspace of G and $U \times I_M$ is one to one on G_1 because $U \times I_M$ is one to one on $A_S \times M$ and $G_1 \subseteq A_S \times M$. Therefore, $U \times I_M$ carries G_1 homeomorphically onto a closed subspace of $(U \times I_M)(G)$. We need only observe that $(U \times I_M)(G_1)$ is dense in $(U \times I_M)(G)$ (remembering that $l_\infty(\Gamma_S)$ has the simple topology) and we have proved that $(U \times I_M)(G_1) = (U \times I_M)(G)$. Since U is a norm isometry on A_S and an algebraic homomorphism it follows from the Stone–Weierstraß theorem that $U(C(K)) = U(A_S)$. Define the projection in the obvious way: $P(f)$ is the unique element of A_S such that $U(P(f)) = U(f)$. Obviously, $P(T) \subseteq T$. Since U is an algebraic homomorphism and $\| U \| = 1$, the remarks above show that this projection is defined by a retraction of K onto $\overline{\Gamma_S}$.

Corollary. *Suppose that $C(K)$ satisfies the hypothesis of the theorem above. Then any separable subspace of K is metrizable; in general,*

$$\text{nden } C(L) = \text{den } L$$

for any compact subset of L [AN].

Proof. This follows from the proof above. Choose any set $E \subseteq K$. Let $\beta = \text{den } E$ be infinite and let F be a dense subset of E that has cardinality no more than β. Let S be any subset of $C(K)$ that separates the points of F and nden $S = \beta$. If we repeat the construction above with the additional requirement that $F \subseteq \Gamma_1$ then the spectrum of A_S, which is the image of the retraction defined above, contains E and nden $A_S = \beta$.

Suppose that we have subsets $S \subseteq S' \subseteq C(K)$ and we construct the spaces $S \subseteq A_S$, the set Γ_S, the operator U and the projection P as above. Suppose that we also construct in an analogous manner spaces $S' \subseteq A_{S'}$, the set $\Gamma_{S'}$, with $\Gamma_S \subseteq \Gamma_{S'}$, the operator V and the projection Q and $A_S \subseteq A_{S'}$. Since the kernel of Q is a subspace of the kernel of P it follows that $PQ = QP = P$. With this observation we may extend the Main Theorem to the following decomposition theorem, which is Vasak's extension of the Amir–Lindenstrauss theorem. It is only a matter of repeating the usual routine details to formulate the following in the case that X is a Fréchet space; we leave this to the interested reader.

THEOREM. *Suppose that X is a Banach space, M is a separable metric space, Φ an s usc compact valued map, with respect to the weak topology, from M into X whose image T spans X (equivalently, separates the points of X^*). Let K be the unit ball of X^* in the weak* topology and consider X as canonically embedded in $C(K)$. We may find an ordinal interval $[1,\beta]$, β is a limit ordinal, and projections $\{P_\alpha : 1 \leqq \alpha \leqq \beta\}$ defined by retractions $\{r_\alpha : 1 \leqq \alpha \leqq \beta\}$ on K, that is $P_\alpha(f) = f \circ r_\alpha$, such that*

(i) $P_1(C(K))$ *is separable*;

(ii) $P_\alpha P_\gamma = P_\gamma P_\alpha = P_{\min\{\alpha,\gamma\}}$;

(iii) P_β *is the identity on* $C(K)$;

(iv) $\operatorname{nden} P_\alpha(C(K)) < \operatorname{nden} C(K)$ *for all* $\alpha < \beta$;

(v) $\operatorname{den} P_\alpha(C(K)) = \operatorname{card}[1,\alpha]$;

(vi) *if γ is a limit ordinal then*

$$P_\gamma(C(K)) = \overline{\bigcup_{\alpha<\gamma} P_\alpha(C(K))} \qquad \textit{and}$$

(vii) $P_\alpha(T) \subseteq T$ *for all α (hence, $P_\alpha(X) \subseteq X$ and each P_α restricted to X is also a projection).*

PROOF. This requires only induction on $\operatorname{nden} X = \beta$ and is obviously true when β is countable. Choose $\{f_\alpha : 1 \leqq \alpha < \beta\} \subseteq T$ that is dense in T. Let $S_\omega = \{f_n : n < \omega\}$ and construct A_ω and Γ_ω as above. In general, let $S_{\alpha+1} = A_\alpha \cup \{f_\gamma : \gamma < \alpha + 1\}$ and assume that $\Gamma_\alpha \subseteq \Gamma_{\alpha+1}$ where these sets are constructed as above. If λ is a limit ordinal, let

$$A_\lambda = \overline{\left(\bigcup_{\alpha<\lambda} A_\alpha\right)} \quad \text{and} \quad \Gamma_\lambda = \bigcup_{\alpha<\lambda} \Gamma_\alpha .$$

The retractions exist as in the Main Theorem.

The result above in the case that $M = \{m\}$ is a one point space and $\Phi(m)$ is a weakly compact subset of X is due to Amir and Lindenstrauss as is the following result.

THEOREM. *If we have the hypothesis of the Main Theorem then there exist a set Δ and an operator $R : C(K) \to c_0(\Delta)$ that is one to one.*

PROOF. Again, this is only induction. Let $\{P_\alpha : 1 \leqq \alpha \leqq \beta\}$ be the projections defined by the retractions $\{r_\alpha : 1 \leqq \alpha \leqq \beta\}$ and let $K_\alpha = r_\alpha(K)$ and $P_0 = 0$. Consider the following operator:

$$R : C(K) \to \left(\sum_\alpha C(K_\alpha)\right)_\infty$$

defined by $R(f) = (P_{\alpha+1} - P_\alpha)f$. Using induction on β we shall prove that the image of R is in $(\Sigma_\alpha C(K_\alpha))_{c_0}$. Suppose this happens for any $\alpha < \beta$. From (vi) of the decomposition this must also happen for β. Observe that the operator R is one to one. The thoerem now follows by induction also. If we assume that there exists an operator $R_\alpha : C(K_\alpha) \to c_0(\Lambda_\alpha)$ that is one to one, and we may assume that $\| R_\alpha \| = 1$, then if we define

$$\sum_\alpha R_\alpha : \left(\sum_\alpha C(K_\alpha)\right)_{c_0} \to \left(\sum_\alpha c_0(\Lambda_\alpha)\right)_{c_0}$$

in the obvious way then $(\Sigma_\alpha R_\alpha) \circ R$ is the desired operator.

The following is elementary.

LEMMA. *Suppose that $T_1, \ldots, T_n$ are usc compact valued images of separable metric spaces. Then the product $\Pi_{1 \leq i \leq n} T_i$ is also the usc compact valued image of a separable metric space and is also Lindelöf.*

The following technique is very old.

LEMMA. *Suppose that $C(K)$ satisfies the hypothesis of the Main Theorem above. Let $E \subseteq K$ and $k_0 \in \bar{E}$. Then there exists a countable $F \subseteq E$ such that $k_0 \in \bar{F}$.*

PROOF. Let $T = \bigcup_m \Phi(m)$ which is Lindelöf in the simple topology as well as T^n for all $n \in \mathcal{N}$. For fixed n and m in $\mathcal{N}$ and $k \in E$ define

$$U_{n,m}(k) = \left\{(f_1, \ldots, f_n) : \sum_{i \leq n} |f_i(k) - f_i(k_0)| < 1/m \text{ and } f_i \in T\right\}.$$

Clearly, this defines a cover of T^n that has a countable subcover

$$\{U(k_{n,m,j}) : j \in \mathcal{N}\}.$$

Since T separates the points of K it follows that $k_0 \in \overline{\{k_{n,m,j} : n, m, j \in \mathcal{N}\}}$.

THEOREM (Gul'ko). *If we have the hypothesis of the Main Theorem then K is a Corson compact (hence, angelic* [Pry]*). In particular, an Eberlein compact is angelic* (see [Ne]).

PROOF. This is also an immediate consequence of the Main Theorem and induction. Assume that, up to a given ordinal β, if we have the decomposition $\{K_\alpha : \omega \leqq \alpha < \beta\}$ then if μ is a measure supported on some K_α then $\{(P^*_{\gamma+1} - P^*_\gamma)\mu \neq 0 : \gamma < \alpha\}$ is countable. If β is the limit of a sequence of smaller ordinals then $\{(P^*_{\gamma+1} - P^*_\gamma)\mu \neq 0 : \gamma < \beta\}$ is countable. Suppose that β is not such an ordinal. Clearly, μ is a cluster point of $\{P^*_\gamma(\mu) : \alpha < \beta\}$, thus a cluster point of a countable subset of $\{P^*_\gamma(\mu) : \alpha < \beta\}$. This means that there exists a $\gamma < \beta$ such that $P^*_\gamma(\mu) = \mu$ and the result follows from the induction hypothesis. The theorem follows easily by induction.

We have not used the definition given in [V] but it is easy to see that our definition is formally more general; indeed, the following shows that these are equivalent. Another consequence of the following Proposition is that the continuous image of a Gul'ko compact is a Gul'ko compact.

PROPOSITION. *Let X be a Banach space and let X_e denote X with the weakest topology such that each extreme point of the unit ball of X^* is continuous on X_e. The following are equivalent*:

(i) *X is weakly countably determined, which means that there exists a sequence $\{K_n : n \in \mathcal{N}\}$ of weak* compact subsets of X^{**} such that for every $x \in X$ there exists a subsequence $\{K_{\xi(n)} : n \in \mathcal{N}\}$ of $\{K_n ; n \in \mathcal{N}\}$ such that $x \in \bigcap_n K_{\xi(n)} \subseteq X$;*

(ii) *there exist a separable metric space M and a map $\Phi : M \to \varphi(X)$ such that $\Phi(F)$ is relatively compact in X_e for any compact subset F of M and $\bigcup_{t \in M} \Phi(t)$ separates the points of X^* and*

(iii) *there exists a separable metric space M and an usc compact (in the weak topology) valued map Φ defined on M such that $X = \bigcup_{t \in M} \Phi(t)$.*

PROOF. If X is weakly countably determined then let

$$M = \left\{ \xi \in \mathcal{N}^{\mathcal{N}} : \bigcap_{n=1}^{\infty} K_{\xi(n)} \subseteq X \right\}.$$

It is completely routine to check that

$$\Phi(\xi) = \bigcap_{n=1}^{\infty} K_{\xi(n)}$$

satisfies (iii) on M. Suppose that we have (iii). Let $\{U_n : n \in \mathcal{N}\}$ be a neighborhood basis of M and define

$$K_{n,p} = \overline{\bigcup_{m \in U_n} \Phi(m) \cap B(0,p)}^{\text{weak}^*}$$

where the closure is taken in X^{**}. For a fixed $x \in X$ it is easy to check that $\bigcap_{x \in K_{n,p}} K_{n,p}$ is actually a subset of X and this verifies (i). Since (iii) implies (ii) is trivial, there remains only to show that (ii) implies (iii). If we have (ii), define

$$T_n = \{m \in M : \Phi(m) \cap B(0,n) \neq \emptyset\},$$

let M_1 be the disjoint union of the sequence $\{T_n\}$ and define $\Phi_1(m) = \Phi(m) \cap B(0,n)$ for $m \in T_n \subseteq M_1$. The version of a theorem of Grothendieck given by Bourgain and Talagrand (see [S] for a proof) says that if F is a compact subset of M_1 then $\Phi_1(F)$ is a relatively weakly compact subset of X. Now, we extend Φ_1 so that we obtain an usc compact valued (in the weak topology) map Φ_2 also defined on M_1 and this is done by defining

$$\Phi_2(t) = \bigcap_{n=1}^{\infty} \bar{c}\left(\bigcup_{d(s,t) \leqq 1/n} \Phi_1(s)\right).$$

It is an easy consequence of the separation theorem that Φ_2 is usc and compact and convex valued in the weak topology. For each finite set of rational numbers $r_1, r_2, \ldots, r_m$ let $M(r_1, r_2, \ldots, r_m)$ be an m-fold product of M_1 and define

$$\Psi(r_1, r_2, \ldots, r_m) : M(r_1, r_2, \ldots, r_m) \to \mathfrak{p}(X)$$

by

$$\Psi(r_1, r_2, \ldots, r_m)(t_1, t_2, \ldots, t_m) = \sum_{i=1}^{m} r_i \Phi_2(t_i).$$

Since the disjoint union T of $\{M(r_1, r_2, \ldots, r_m)\}$ is also a separable metric space we are now in a position to assume that Ψ is usc compact valued and $\bigcup_{t \in M} \Psi(t)$ is dense in X. Define

$$T_n = \{t \in T : \Psi(t) \cap \bar{B}(0, 2^{-n}) \neq \emptyset\}$$

and define

$$\Psi_n(t) = \Psi(t) \cap \bar{B}(0, 2^{-n}).$$

The desired map is

$$\prod_n \Psi_n : \prod_n T_n \to \mathfrak{p}(X)$$

where

$$\prod_n (\Psi_n)(t_n) = \sum_n \Psi_n(t_n).$$

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